Math 246A Lecture 7 Notes

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1 Complex Projective Space and Introduction to Cauchy's Theorem

1.1 Complex projective space

Definition 1.1. Let $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. Then **complex projective space** $\mathbb{C}P^1$ is the set of complex lines in $\mathbb{C}^2$.

We can think of this as

$$\left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} : |z_1|^2 + |z_2|^2 \neq 0 \right\} / \cong,$$

where $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \cong \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $w_1 = \lambda z_1$. Now let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 complex matrix with $ad - bc \neq 0$. Then $A : \mathbb{C}P^1 \to \mathbb{C}P^1 \iff \det(A) \neq 0$.

Given A define $T_A : \mathbb{C}P^1 \to \mathbb{C}P^1$ as $T_A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. For all A and A', A and A' have the same action on $\mathbb{C}P^1$ if and only if $A' = \lambda A$. So

$$\mathrm{PSL}(2, \mathbb{C}) = \{A : \det A = 1\}/(\pm I).$$

Given $L, M \in \mathbb{C}P^1$ there exists an $A \in \mathrm{PSL}(2, \mathbb{C})$ such that $A(L) = M$. Then

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{bmatrix} \iff \frac{w_1}{w_2} = \frac{az_1/z_2 + b}{cz_1/z_2 + d}.$$

We therefore get a correspondence between complex projective space, $\mathbb{C}^*$, and the Riemann sphere.

$\mathbb{C}P^1$	$\mathbb{C}^*$	$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$
$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ $\big\downarrow T_A$ $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$	$z = z_1/z_2$ $\big\downarrow T_A$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ $w = \frac{az+b}{cz+d}$	$\frac{x_1+ix_2}{1-x_3} = z$ $\big\downarrow T_A$ $\frac{y_1+iy+2}{1-y_3}$

Let's look at the generators and their action on S^2.

1. $z \mapsto e^{i\theta} z$: This fixes the north and south poles, and rotates the sphere by the angle θ about the vertical axis. We get $y_3 = x_3$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

2. $z \mapsto kz$, $k > 0$: If $k > 1$, horizontal circles get shifted up the sphere. If $k < 1$, horizontal circles get shifted down the sphere, like a flow.

3. $z \mapsto 1/z$: This flips the Riemann sphere upside down.

4. $z \mapsto z + \lambda$, $\lambda = \mathbb{C}$: Without loss of generality, $\lambda = 1$. Look at what this does to lines in $\mathbb{C}^*$, which are circles on S^2 that pass through the north pole.

1.2 Cauchy's theorem, simplest form

Definition 1.2. A C^1 **parameterized curve** is a function $\gamma = \{z(t) : z : [a, b] \to \mathbb{C}\}$ such that z is C^1 on $(a - \varepsilon, b + \varepsilon)$ for some $\varepsilon > 0$.

Definition 1.3. The **contour integral** over γ is

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t)) z'(t)\, dt.$$

Now suppose $h : [c, d] \to [a, b]$ is C^1 and increasing. Then by change of variables,

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t)) z'(t)\, dt = \int_c^d f(z(h(s))) z'(h(s)) h'(s)\, ds = \int_c^d f(w(s)) w'(s)\, ds,$$

where $w(s) = z(h(s))$. If h is decreasing, $h(c) = b$, and $h(d) = a$, then

$$\int_c^d f(w(s)) w'(s)\, ds = \int_a^b f(z(t)) z'(t)\, dt.$$

If $\gamma = \gamma_1 + \cdots + \gamma_n$ and $\gamma_j[a_j, b_j] \to \mathbb{C}$ is C^1 with $b_j = a_{j+1}$ and $\gamma_j(b_j) = v - j + 1(a_{j+1})$, then

$$\gamma(t) = \sum_j \gamma_j(t) \mathbb{1}_{[a_j, b_j]}$$

is piecewise C^1. If f is continuous on γ, define

$$\int_\gamma f(z)\, dz = \sum_j \int_{\gamma_j} f(z)\, dz.$$

Lemma 1.1.

$$\left| \int_\gamma f(z)\, dz \right| \leq (\sup_\gamma |f(z)|) \int_a^b |z'(t)|\, dt.$$

Proof.

$$\left| \int_a^b f(z) z'(t)\, dt \right| \leq \int_a^b |f(z)||z'(t)|\, dt \leq (\sup_\gamma |f(z)|) \int_a^b |z'(t)|\, dt. \qquad \square$$

Observe that $\int_a^b |z'(t)|\, dt$ is the length of the parametrized curve γ.

Definition 1.4. A **domain** (or a region) $\Omega \subseteq \mathbb{C}$ is a connected open set.

Definition 1.5. $H(\Omega)$ is the set of functions $f : \Omega \to \mathbb{C}$ such that f' exists and is continuous on Ω.

Definition 1.6. $A(\Omega)$ is the set of functions $f : \Omega \to \mathbb{C}$ such that for all $z_0 \in \Omega$, there is a $\delta(z_0) > 0$ usch that if $B = \{|z - z_0| < \delta(z_0)\} \subseteq \Omega$, then

$$f(z) = \sum_{n-0}^{\infty} a_n(z_0)(z - z_0)^n$$

in B.

These are the functions with convergent power series in Ω.

Lemma 1.2. *Let $f \in H(\Omega)$, γ be piecewise C^1, and $\gamma \subseteq \Omega$. Then*

$$\int_\gamma f'(z)\, dz = f(\gamma(b)) - f(\gamma(a)).$$

Proof.

$$\int_a^b f'(z(t)) z'(t)\, dt = \int_a^b \frac{d}{dt} f(z(t))\, dt = f(\gamma(b)) - f(\gamma(a)). \qquad \square$$

Let R be a rectangle in $\mathbb{C}$ with vertices $a+ic, b+ic, a+id, b+id$, where $a<b$, $c<d$, and $a,b,c,d \in \mathbb{R}$. Then $\partial R = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where

$$\gamma_1 = (a+ci) + t(b-a) \qquad 0 \le t \le 1,$$

$$\gamma_2 = (b+ci) + ti(d-c) \qquad 0 \le t \le 1,$$

and so on.

Theorem 1.1 (Cauchy integral formula for a rectangle)**.** *If* $\overline{R} \subseteq \Omega$ *and* $f \in H(\Omega)$*, then*

$$\int_{\partial R} f(z)\, dz = 0.$$

Proof.

$$\begin{aligned}\int_{\partial R} f(z)\, dz &= \int_a^b f(z+ic)\, dx + i\int_c^d f(b+iy)\, dy - \int_a^b f(x+id)\, dx - i\int_c^d f(a+iy)\, dt \\ &= I + II + III + IV\end{aligned}$$

$$I + III = \int_a^b (f(z+ic) - f(z+id))\, dx = \int_a^b \left(-\int_c^d \frac{\partial f}{\partial y}(x+iy)\, dy \right) dx$$

$$II + IV = \int_c^d \int_a^b \frac{\partial f}{\partial x}(x+iy)\, dy\, dx$$

But $\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial x} = 0$ by the Cauchy-Riemann equations. $\square$